

Algebraic Quantum Theory on Manifolds:
A Haag-Kastler Setting for Quantum Geometry

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May 31, 1999

Abstract

Motivated by the invariance of current representations of quantum gravity under diffeomorphisms much more general than isometries, the Haag-Kastler setting is extended to manifolds without metric background structure. First, the causal structure on a differentiable manifold M of arbitrary dimension ($d + 1 > 2$) can be defined in purely topological terms, via cones (C-causality). Then, the general structure of a net of C^* -algebras on a manifold M and its causal properties required for an algebraic quantum field theory can be described as an extension of the Haag-Kastler axiomatic framework.

An important application is given with quantum geometry on a spatial slice Σ within the causally exterior region of a topological horizon \mathcal{H} , resulting in a net of Weyl algebras for states with an infinite number of intersection points of edges and transversal $(d - 1)$ -faces within any neighbourhood of the spatial boundary $\mathcal{H} \cap \Sigma \cong S^2$.

1 Introduction

While classical general relativity usually employs a Lorentzian spacetime structure, the most successful approaches for quantum gravity, such as the canonical quantization of the connection representation, the loop representation or the spin network representation, and topological quantum field theory, BF-theories and spin foams are invariant under diffeomorphism much more general than isometries. Currently, most of these approaches are mainly proposals for the quantum theory of some free geometry. Although in particular situations also coupling to matter has been studied, at present in spacetime dimension $3 + 1$ no complete theory for quantum geometry coupled to quantum matter is available. This situation is remarkably paralleled by the algebraic framework of quantum field theory (QFT) on a curved spacetime of dimension $3 + 1$, which is most successful in describing the structure of free or asymptotically free quantum field theories. However, the standard Haag-Kastler quantum field theories over Minkowski space or curved asymptotically free spacetimes is not invariant under diffeomorphisms unless these are isometries.

Recently a diffeomorphism invariant extension of the axiomatic Osterwalder-Schrader framework [1] of constructive Euclidean QFT [2] has been given by [3]. There the Osterwalder-Schrader reconstruction of Euclidean QFT was generalized. However there is no obvious generalization of those analyticity properties which relate the Schwinger functions and Wightman functions of the standard Euclidean and Minkowskian QFT. At present there is no notion of a generalized "Wick rotation" which would allow us to use the results of [3] in order to infer properties of an appropriate diffeomorphism invariant generalization of the algebraic Haag-Kastler framework for Lorentzian QFT.

Recently the algebraic framework was used successful in providing a clear proof [4] of the holographic hypothesis (Maldacena conjecture [5, 6]). While there is no direct point transformation between the bulk degrees of freedom in anti-de Sitter (AdS) space and those of the boundary CFT, the relation becomes clear for the corresponding algebras localized in wedge regions on AdS space and in double cones on its boundary.

Therefore it is the goal of the present investigations to extend the framework of algebraic quantum field theory (AQFT) and Haag-Kastler axioms such that it becomes also applicable to theories which are invariant under a larger class of diffeomorphisms, such as quantum gravity. Within the AQFT setting the question rises what are the general classes of diffeomorphisms compatible with a given algebraic structure.

Earlier attempts [7, 8, 9] towards a diffeomorphism invariant algebraic setting for quantum field theory mainly generalized those axioms that use only a topological structure on the manifold already in the usual setting, such as isotony and covariance. However the causality axiom could only be formulated in a very rudimentary sense, namely by generalizing Haag duality just on the boundary of the net. That procedure introduced quite strange features on the net of $*$ -algebras. In particular it implied the existence of an Abelian center in the localized algebras which was then associated with the minimal (interior) boundary [7]. The existence of such an Abelian center implied in particular that the algebras could not be usual CCR Weyl algebras, since these would have to be simple if the symplectic form was nondegenerate.

Nevertheless, presently it is known that the algebraic structure of a free quantum field theory on Minkowski space or an asymptotically free field theory on a (usually globally

hyperbolic) curved space-time is encoded in a *causal* net of C^* -algebras. In particular on Minkowski space, there exists a strong correspondence between particular causal sets of Minkowski space and localized C^* -algebras of the net. There are particular causal sets which form a topological basis of Minkowski space, namely the bounded double cones. Moreover, for a net of subalgebras of a *Weyl* algebra, it is possible [10] to work with a flexible notion of causality rather than with a rigidly given one, and in principle the net together with its underlying manifold might be reconstructed from the relation among the localized C^* -algebras only [11].

This motivates the present approach where we present a generalization of this causal net structure in an a priori background independent ("diffeomorphism invariant") manner. Here the net has to be background independent, but still compatible with a (metric independent) notion of causality. In order to achieve this, we have to abstract an appropriate *topological* notion of causality. Appropriate definitions for such a notion were given recently in [12]. Those diffeomorphisms which preserve such topological causal structure, should naturally also leave invariant the algebraic structure of the net. A causal topology on a $d+1$ -dimensional manifold then provides a topological notion of a causal complement on any set. The next step is then to find a natural implementation of the correspondence between causal sets on a differentiable manifold and C^* -algebras localized on these sets. This amounts to define a causal net of C^* -algebras on causal differentiable manifolds.

The first condition which a causal net of C^* -algebras over a causal topological space should naturally satisfy is that C^* -algebras over causally disjoint sets mutually commute. Note that e.g. in a $d+1$ -dimensional black hole spacetime, the intersection of a spatial slice with the horizon is a $d-1$ -dimensional sphere S^{d-1} . The latter may be viewed as the boundary of a minimal d -dimensional open set O_{\min} contained in any larger d -dimensional open set O_{\max} within a spatial slice which extends through all of the region exterior of the horizon up to spatial infinity i^0 . In [7] a generalization of Haag duality was implemented algebraically, by demanding that the commutant of the (asymptotic) global algebra equals a minimal Abelian center algebra located over the minimal set.

The possible results of a concrete observation are encoded in a corresponding state on the causal net of C^* -algebras. Particularly convenient states for quantum geometry are the spin-network states. A state introduces additional structure which may serve to distinguish gauge invariances from more genuine symmetries (like unitarity of the dynamics). Given a causal foliation of spatial slices exterior to a topological horizon (via causal boundaries), those diffeomorphisms which leave invariant the causal foliation are purely gauge.

In quantum geometry spin-network states are given via an embedding of a closed graph into a particular slice of the foliation. Then there exists diffeomorphisms which keep the set of all slices invariant but change the foliation monotonously, preserving the natural order of the slices. These are topological dilations. If there was no state two foliations related by such a change should be considered as equivalent. However an embedded graph can eventually detect a monotonous change in the foliation by a change in the original relations between the slices and the graph, which are given by the intersections of the edges of the graph with the slices of the foliation. The relations between an embedded graph and a foliation are encoded topologically in the intersections of edges with slices of a foliations. Change of this intersection topology by dilating one slice onto another can

result in changes of the C^* -algebra. Therefore dilation diffeomorphisms can not be gauge here, but rather should correspond to outer (i.e. non-trivially represented) isomorphisms on the algebras.

2 Cone causality on differentiable $d + 1$ -manifolds

Let us now define the notion of a causal topology for general differentiable $d + 1$ -manifold M within any differentiability category which unless specified otherwise should not be larger than C^1 and for convenience may be taken C^∞ . (With minor modifications an extension to the C^0 category is possible too [12]. However for the present purpose the differentiable setting is most convenient.) Let

$$\mathcal{C} := \{x \in \mathbb{R}^{d+1} : x_0^2 = (x - x_0 e_0)^2\}, \mathcal{C}^+ := \{x \in \mathcal{C} : x_0 \geq 0\}, \mathcal{C}^- := \{x \in \mathcal{C} : x_0 \leq 0\} \quad (2.1)$$

be the standard (unbounded double) light cone, and the forward and backward subcones in \mathbb{R}^{d+1} , respectively. The standard open interior and exterior of \mathcal{C} is defined as

$$\mathcal{T} := \{x \in \mathbb{R}^{d+1} : x_0^2 > (x - x_0 e_0)^2\}, \mathcal{E} := \{x \in \mathbb{R}^{d+1} : x_0^2 < (x - x_0 e_0)^2\}. \quad (2.2)$$

A *manifold thickening* with thickness $m > 0$ is given as

$$\mathcal{C}^m := \{x \in \mathbb{R}^{d+1} : |x_0^2 - (x - x_0 e_0)^2| < m^2\}, \quad (2.3)$$

The characteristic topological data of the standard cone is encoded in the topological relations of all its manifold subspaces (which includes in particular also the singular vertex O) and among each other.

Definition 1: Let M be a differentiable manifold. A (*null*) *cone* at $p \in \text{int} M$ is the diffeomorphic image $\mathcal{C}_p := \phi_p \mathcal{C}$ of a diffeomorphism of topological spaces $\phi_p : \mathcal{C} \rightarrow \mathcal{C}_p \subset M$ with $\phi_p(0) = p$, such that

- (i) every (differentiable) submanifold $N \subset \mathcal{C}$ is mapped diffeomorphically on a submanifold $\phi_p(N) \subset M$,
- (ii) for any two submanifolds $N_1, N_2 \subset \mathcal{C}$ there exist diffeomorphisms $\phi_p(N_1) \cap \phi_p(N_2) \cong N_1 \cap N_2$ and $\phi_p(N_1) \cup \phi_p(N_2) \cong N_1 \cup N_2$,
- and (iii) for any two differentiable curves $c_1, c_2 :] - \epsilon, \epsilon[\rightarrow \mathcal{C}$ with $c_1(0) = c_2(0) = p$ it holds $T_0 c_1 = T_0 c_2 \Leftrightarrow T_p(\phi_p \circ c_1)|_{]-\epsilon, \epsilon[} = T_p(\phi_p \circ c_2)|_{]-\epsilon, \epsilon[}$.

Condition (iii) says that the well defined notion of transversality of intersections at the vertex is preserved by ϕ_p .

Definition 2: An (*ultralocal*) *cone structure* on M is an assignment $\text{int} M \ni p \rightarrow \mathcal{C}_p$ of a cone at every $p \in \text{int} M$.

Note that, although $\mathcal{C}_p = \phi_p(\mathcal{C})$, \mathcal{T} and \mathcal{E} here need not be diffeomorphic to $\phi_p(\mathcal{T})$ and $\phi_p(\mathcal{E})$ respectively. A cone structure on M can in general be rather wild with cones at different points totally unrelated unless we impose a topological connection between the cones at different points. The cone structure can be tamed by the requirement of differentiability of the family $\{p \rightarrow \mathcal{C}_p\}$.

Definition 3: Let M be a differentiable manifold. A *weak (\mathcal{C}) local cone (LC) structure* on M is a cone structure which is differentiable i.e. $\{p \rightarrow \mathcal{C}_p\}$ is a differentiable family.

A weak LC structure at each point $p \in \text{int}M$ defines a characteristic topological space \mathcal{C}_p of codimension 1 which is Hausdorff everywhere but at p . In particular \mathcal{C}_p does not contain any open $U \ni p$ from the manifold topology of M . This can be improved by resolving the cone.

Definition 4: Let M be a differentiable manifold. A *(manifold) thickened cone* of thickness $m > 0$ at $p \in \text{int}M$ is the diffeomorphic image $\mathcal{C}_p^m := \phi_p \mathcal{C}^m$ of a diffeomorphism of manifolds $\phi_p : \mathcal{C} \rightarrow \mathcal{C}_p \subset M$ with $\phi_p(0) = p$.

Note that due to the manifold property a thickened cone is much more simple than a cone itself.

Definition 5: A *thickened cone structure* on M is an assignment $\text{int}M \ni p \mapsto \mathcal{C}_p^{m(p)}$ of a thickened cone at every $p \in \text{int}M$.

Note that in general the thickness m can vary from point to point in M . However it is natural to require $m : M \rightarrow \mathbb{R}_+$ to be differentiable.

Definition 6: Let M be a differentiable manifold. A *strong (\mathcal{C}^m) LC structure* on M is a differentiable family of diffeomorphisms $\phi_p : \mathcal{C}^m \rightarrow \mathcal{C}_p^{m(p)} \subset M$ with $\phi_p(0) = p$ and such that the thickness m is a differentiable function on M .

Note that a strong LC structure is still much more flexible than a conformal structure. For any $q \neq p$ the tangent directions given by $T_q \mathcal{C}_p$ need a priori not be related to tangent directions of null curves of g , since the cone (or its thickening) at p is in general unrelated to that at q .

Theorem 1: Let M carry a strong LC structure. At any $p \in \text{int}M$ there exists an open $U \ni p$ such that:

For $d := \dim M - 1 > 0$ it is $|\Pi_0(\mathcal{T}_p|_U)| = 2$ and $\Pi_{d-1}(\mathcal{E}_p|_U) = \Pi_{d-1}(S^{d-1})$,

for $d > 1$ it is $\Pi_{d-1}(\mathcal{T}_p|_U) = 0$ and $|\Pi_0(\mathcal{E}_p|_U)| = 1$,

for $d = 1$ it is $\Pi_{d-1}(\mathcal{T}_p|_U) = \Pi_{d-1}(\mathcal{E}_p|_U) = \Pi_0(S^0)$, i.e. $|\Pi_0(\mathcal{T}_p|_U)| = |\Pi_0(\mathcal{E}_p|_U)| = 2$,

and in dimension $d = 0$ it is $\mathcal{T}_p = \mathcal{E}_p = \emptyset$.

Proof: For all $p \in \text{int}M$ the strong LC structure provides a thickened cone $\mathcal{C}_p^{m(p)}$. Since $m(p) > 0$, $\mathcal{C}_p^{m(p)}$ contains always a neighborhood $U \ni p$ diffeomorphic to a neighborhood $\phi_p^{-1}(U) \ni 0$ of the standard cone which in any dimension has the desired properties. \square

At any interior point $p \in \text{int}M$ the open exterior \mathcal{E}_p and the open interior \mathcal{T}_p of the cone \mathcal{C}_p are locally topologically distinguishable for $d > 1$, indistinguishable for $d = 1$, and empty for $d = 0$. With a strong LC structure, $\mathcal{T}_p|_U \neq \mathcal{E}_p|_U \forall U \ni p \iff d + 1 > 2$, whence locally in any neighborhood $U \ni p$ the interior and exterior of $\mathcal{C}_p \cap U$ at p in U has an intrinsic invariant meaning. $\mathcal{C}_p|_U$ divides $U - \mathcal{C}_p|_U$ in three open submanifolds, a non-contractable exterior $\mathcal{E}_p|_U$, plus two contractable connected components of $\mathcal{T}_p =: \mathcal{F}_p|_U \cup \mathcal{P}_p|_U$, the local future $\mathcal{F}_p|_U$ and the local past $\mathcal{P}_p|_U$ with $\partial \mathcal{F}_p|_U = \mathcal{C}_p^+|_U$ where $\mathcal{C}_p^+ := (\phi_p \mathcal{C}^+)$ and $\partial \mathcal{P}_p|_U = \mathcal{C}_p^-|_U$ where $\mathcal{C}_p^- := \phi_p \mathcal{C}^-$ respectively. This rises also the question if and how \mathcal{F}_p and \mathcal{P}_p or their local restriction to $U \ni p$ can be distinguished. This problem is solved by a topological \mathbb{Z}_2 connection.

Let M be differentiable and τ be any vector field $M \rightarrow TM$ such that at any $p \in \text{int}M$ its orientation agrees with that of $\phi_p(a)$. Such a orientation vector field comes naturally along with a \mathbb{Z}_2 -connection on M which allows to compare the orientation $\tau(p)$ at different $p \in \text{int}M$. Given a strong LC structure on M , the \mathbb{Z}_2 -connection is in fact provided via continuity of $p \mapsto T_p \phi_p(a)$. Then τ is tangent to an integral curve segment through p from

\mathcal{P}_p to \mathcal{F}_p . In particular, \mathcal{F}_p and \mathcal{P}_p are distinguished from each other by a consistent τ -orientation on M .

In order to obtain a causal structure it remains to identify natural consistency conditions for the intersections of cones of different points. In order to define topologically timelike, lightlike, and spacelike relations, and a reasonable causal complement, we introduce the following causal consistency conditions on cones.

Definition 7: M is (locally) *cone causal* or *C-causal* in an open region U , if it carries a (weak or strong) LC structure and in U the following relations between different cones in $\text{int}M$ hold:

- (1) For $p \neq q$ one and only one of the following is true:
 - (i) q and p are causally *timelike* related, $q \ll p : \Leftrightarrow q \in \mathcal{F}_p \wedge p \in \mathcal{P}_q$ (or $p \ll q$)
 - (ii) q and p are causally *lightlike* related, $q \triangleleft p : \Leftrightarrow q \in \mathcal{C}_p^+ - \{p\} \wedge p \in \mathcal{C}_q^- - \{q\}$ (or $p \triangleleft q$),
 - (iii) q and p are causally unrelated, i.e. relatively *spacelike* to each other, $q \bowtie p : \Leftrightarrow q \in \mathcal{E}_p \wedge p \in \mathcal{E}_q$.
- (2) Other cases (in particular non symmetric ones) do not occur.

M is *locally C-causal*, if it is C-causal in any region $U \subset M$. M is *C-causal* if conditions (1) and (2) hold $\forall p \in \mathcal{C}$.

Let M be C-causal in U . Then, $q \ll p \Leftrightarrow \exists r : q \in \mathcal{P}_r \wedge p \in \mathcal{F}_r$, and $q \triangleleft p \Leftrightarrow \exists r : q \in \mathcal{C}_r^+ \wedge p \in \mathcal{C}_r^-$.

If an open curve $\mathbb{R} \ni s \mapsto c(s)$ or a closed curve $S^1 \ni s \mapsto c(s)$ is embedded in M , then in particular its image is $\text{im}(c) \equiv c(\mathbb{R}) \cong \mathbb{R}$ or $\text{im}(c) \equiv c(S^1) \cong S^1$ respectively, whence it is free of selfintersections. Such a curve is called *spacelike* : $\Leftrightarrow \forall p \equiv c(s) \in \text{im}(c) \exists \epsilon : c|_{[s-\epsilon, s+\epsilon] - \{0\}} \in \mathcal{E}_{c(s)}$, and *timelike* : $\Leftrightarrow \forall p \equiv c(s) \in \text{im}(c) \exists \epsilon : c|_{[s-\epsilon, s+\epsilon] - \{0\}} \in \mathcal{F}_{c(s)}$.

Note that C-causality of M forbids a multiple refolding intersection topology for any two cones with different vertices. In particular along any timelike curve the future/past cones do not intersect, because otherwise there would exist points which are simultaneously timelike and lightlike related. Continuity then implies that future/past cones in fact foliate the part of M which they cover. Hence, if there exists a fibration $\mathbb{R} \hookrightarrow \text{int}M \rightarrow \Sigma$, then C-causality implies in particular that the future/past cones foliate on any fiber. Therefore C-causality allows a reasonable definition of a causal complement.

Definition 8: A *causal complement* in a set U is a map $P(U) \ni S \mapsto S^\perp \in P(U)$ such that

- (i) $S \subseteq S^{\perp\perp}$
- (ii) $(\bigcup_j S_j)^\perp = \bigcap_j (S_j)^\perp$
- (iii) $S \cap S^\perp = \emptyset$

Example 1: For any open set S in a C-causal manifold M the *causal complement* is defined as

$$S^\perp := \bigcap_{p \in \text{cl}S} \mathcal{E}_p, \quad (2.4)$$

where $\text{cl}S$ denotes the closure in the topology induced from the manifold. Although the causal complement is always open, it will in general not be a contractable region even if S itself is so.

Assume p and q are timelike related, $p \in \mathcal{P}_q$ and $q \in \mathcal{F}_p$. $\mathcal{K}_p^q := \mathcal{F}_p \cap \mathcal{P}_q$ is the bounded open double cone between p and q . Given any bounded open \mathcal{K} such that

$\exists p, q \in M : \mathcal{K} = \mathcal{F}_p \cap \mathcal{P}_q$, we set $i^+(\mathcal{K}) := \{q\}$, $i^-(\mathcal{K}) := \{p\}$, and $i^0(\mathcal{K}) := \mathcal{C}_p^+ \cap \mathcal{C}_q^-$. For any $\mathcal{K}_p^q \subset M$ let $\text{cl}_c(\mathcal{K}_p^q) := \mathcal{K}_p^q \cup [(\partial \mathcal{F}_p \cup \partial \mathcal{P}_q) \cap \partial \mathcal{K}_p^q]$ be its *causal closure*.

Since C-causality prohibits relative refolding of cones, it also ensures that $(\mathcal{K}_p^q)^{\perp\perp} = \mathcal{K}_p^q$, i.e. double cones are closed under $(\cdot)^{\perp\perp}$.

3 Causal index sets and diffeomorphisms

Let us now define the index sets which will be used in our nets of algebras. The natural numbers \mathbb{N} are the most common index set for any countable set on which they induce then a canonical order relation. However, in the following we consider more general index sets which need not be countable.

Definition 9: A *net index set* is an index set I (i) with a partial order \leq , (ii) with a sequence of $K_i \in I$, $i \in \mathbb{N}$, such that $\forall K \in I \exists j \in \mathbb{N} : K \leq K_j$, and (iii) such that each bounded $J \subset I$ has a unique supremum $\sup J \in I$.

Remark 1: If I is totally ordered (iii) is satisfied trivially.

Remark 2: By (ii) a net index set is infinite unless $\exists j \in \mathbb{N} : K_i = K_j \forall i \geq j$.

Definition 10: A *causal disjointness relation* in a net index set I is a symmetric relation \perp such that

- (i) $K_1 \perp K_0 \wedge K_2 < K_1 \Rightarrow K_2 \perp K_0$,
- (ii) for any bounded $J \subset I : K_0 \perp K \forall K \in J \Rightarrow K_0 \perp \sup J$,
- (iii) $\forall K_1 \in I \exists K_2 \in I : K_1 \perp K_2$.

A *causal index set* (I, \perp) is a net index set with a causal disjointness relation \perp .

Definition 11: Let M be infinite with causal complement \perp . M is \perp -*nontrivially inductively covered*, iff \exists sequence of nonempty $K_i \subset M$, $i \in \mathbb{N}$, mutually different with $(K_i)^\perp \neq \emptyset$ such that $\bigcup_{i=1}^\infty K_i = M$.

Example 2: Any conformal class of a Lorentzian metric, which is globally hyperbolic without any singularities determines such a causal structure. In this case the compact open double cones form a basis of the usual Euclidean $d+1$ topology. Each open compact double cone \mathcal{K} is conformally equivalent to a copy of Minkowski space. Consider a spatial Cauchy section Σ of M and a geodesic world line $p : \tau \rightarrow M$ intersecting Σ at $p(0)$, where τ is the proper time of the observer. Now for any $\tau > 0$ the causal past of $p(\tau)$ intersects Σ in an open set O_τ . Then these open sets are totally ordered by their nested inclusion in Σ , and their order agrees also with the total order of worldline proper time,

$$O_{\tau_1} \subset O_{\tau_2} \Leftrightarrow \tau_1 < \tau_2. \quad (3.5)$$

This is the motivation to consider the partial order related to the flow of time and the one related to enlargement in space to be essentially the same, such that in the absence of an a priori notion of a metric time, the nested spatial inclusion will provide a partial order substituting time. (Of course this is in essence similar to the old idea in cosmology of time given by the volume of a closed, expanding universe.)

Consider now a double cone \mathcal{K} in M with $O := \mathcal{K} \cap \Sigma$ and $\partial O = i^0(\mathcal{K})$ and a diffeomorphism ϕ in M with pullbacks $\phi^\Sigma \in \text{Diff}(\Sigma)$ to Σ and $\phi^\mathcal{K} \in \text{Diff}(\mathcal{K})$ to \mathcal{K} . If $\phi(\mathcal{K}) = \mathcal{K}$, it can be naturally identified with an element of $\text{Diff}(\mathcal{K})$. ($\phi = \text{id}_{M-\mathcal{K}}$ is a

sufficient but not necessary condition for that to be true.) If in addition $\phi(\Sigma) = \Sigma$ then also $\phi(O) = O$, and $\phi|_O$ is a diffeomorphism of O .

Let us now consider a 1-parameter set of double cones $\{\mathcal{K}_p\}$ sharing 2 common null curve segments $n_{\pm} \in \partial\mathcal{K}_p$ from $i^{\pm}(\mathcal{K}_p)$ respectively to $i \in i^0(\mathcal{K}_p)$ which they intersect transversally in Σ . Let such cones be parametrized by a line c in Σ starting (transversally to n) at i to some endpoint f on $\partial\Sigma$ (at spatial infinity) such that p is an interior point of $O_p := \mathcal{K}_p \cap \Sigma$. Then we call the limit $W(n_{\pm}, c) := \lim_{p \rightarrow f} K_p$ the wedge in the surface through n_{\pm} and c . Note that in the usual (say Minkowski) metric case a wedge has a quite rigid structure, because c has a canonical location in a surface spanned by n_{\pm} . The present diffeomorphism invariant analogue is of course much less unique in structure.

4 Axioms for QFT on manifolds

Clearly QFT on a globally hyperbolic space-time manifold satisfies isotony (N1), covariance (N2), causality (C), additivity (A) and existence of a (state dependent GNS) vacuum vector (V). More particular on Minkowski space there is a unique Poincare-invariant state ω such that there is a translational subgroup of isometries with spectrum in the closure of the forward light cone only. However there is no reason to expect such features in a more general context. However, an invariant GNS vacuum vector Ω still exists for a globally hyperbolic space-time, although in general it depends on the choice of the state ω . Hence we will now generalize the axioms of AQFT from globally hyperbolic space-times to differentiable manifolds.

For a given QFT on manifolds, say the example of quantum gravity examined below, it remains to check which of the generalized axioms will hold true.

4.1 General axioms for QFT on a differentiable manifold

On a differentiable manifold M part of the AQFT structure can be related to the topological structure of M . The following AQFT axioms are purely topological and should hold on arbitrary differentiable manifolds. Let M be a differentiable manifold with additional structure s (which may be empty) and $\text{Diff}(M, s)$ denote all diffeomorphisms which preserve s . A $\text{Diff}(M, s)$ -invariant algebraic QFT (in the state ω) can be formulated in terms of axioms on a net of $*$ -algebras $\mathcal{A}(\mathcal{O})$ (together with a state ω thereon). It should at least satisfy the following axioms:

N1 (Isotony):

$$\mathcal{O}_1 \subset \mathcal{O}_2 \quad \Rightarrow \quad \mathcal{A}(\mathcal{O}_1) \subset \mathcal{A}(\mathcal{O}_2) \quad \forall \mathcal{O}_{1,2} \in \text{Diff}(M, s) \quad (4.1)$$

N2 (Covariance):

$$\text{Diff}(M, s) \ni g \mapsto U(g) \in U(\text{Diff}(M, s)) \quad : \quad \mathcal{A}(g\mathcal{O}) = U(g)\mathcal{A}(\mathcal{O})U(g)^{-1} . \quad (4.2)$$

(N1) and (N2) are purely topological, involving only the mere definition of the net. These axioms make sense even without a causal structure (see also [7]).

If $\mathcal{A}(\mathcal{O})$ is a C^* -algebra with norm $\|\cdot\|$, it makes sense to impose the following additional axioms:

A (additivity):

$$\mathcal{O} = \cup_j \mathcal{O}_j \quad \Rightarrow \quad \mathcal{A}(\mathcal{O}) = \text{cl}_{\|\cdot\|} (\cup_j \mathcal{A}(\mathcal{O}_j)) . \quad (4.3)$$

V (Invariant Vacuum Vector): Given a state ω , there exists a representation π_ω on a Hilbert space \mathcal{H}_ω such that

$$\begin{aligned} \exists \Omega \in \mathcal{H}_\omega, \|\Omega\| = 1 \quad : \\ \text{(cyclic)} \quad & (\cup_{\mathcal{O}} \mathcal{R}(\mathcal{O})) \Omega \stackrel{\text{dense}}{\subset} \mathcal{H}_\omega \\ \text{(invariant)} \quad & U(g)\Omega = \Omega, \quad g \in \text{Diff}(M, s) . \end{aligned} \quad (4.4)$$

Note: For any $*$ -algebra, the representation π_ω is given by the GNS construction, \mathcal{H}_ω is the GNS Hilbert space. Properties of Ω are induced by corresponding properties of the state ω . The main issue to check is the invariance under a unitary representation U of $\text{Diff}(M, s)$.

4.2 Axioms for QFT on a manifold with cone causality

With a notion of causality on a differentiable manifold M as defined in the previous section, the algebraic structure of a QFT should be related to the causal differential structure of M by further axioms abstracted from the space-time case. In this case it is natural to consider nets of von Neumann algebras. On a causal differential manifold M (in the sense defined above) the algebraic structure of a QFT should satisfy the following axioms which require the notion of a causal complement. Let M be a causal differentiable manifold with additional structure s (which may be empty) and $\text{Diff}(M, s)$ denote all differentiable diffeomorphisms which preserve s , where s is at least a causal structure, eventually with some additional structure s' . A $\text{Diff}(M, s)$ -invariant algebraic QFT in the state ω is a net of von Neumann-algebras $\mathcal{R}(\mathcal{O})$ with a state ω satisfying the following axioms:

C (causality):

$$\mathcal{O}_1 \perp \mathcal{O}_2 \quad \Rightarrow \quad \mathcal{R}(\mathcal{O}_1) \subset \mathcal{R}(\mathcal{O}_2)' . \quad (4.1)$$

CA (causal additivity):

$$\mathcal{O} = \cup_j \mathcal{O}_j \quad \Rightarrow \quad \mathcal{R}(\mathcal{O}) = (\cup_j \mathcal{R}(\mathcal{O}_j))'' . \quad (4.2)$$

Remarks: In the case that the net has both inner and exterior boundary, (4.1) had been weakened in [7] to a generalization of Haag duality on the boundary of the net. Here we do not assume a priori the existence of such a boundary of the net. However an example of quantum geometry with such a boundary structure is discussed below.

Given a net of C^* algebras consistent with a norm $\|\cdot\|$, it made sense to impose (A) above. If the algebras are in particular also von Neumann ones (A) should be sharpened

to (CA). In the general case of $*$ -algebras (not necessarily C^* ones) the algebraic closure has no natural topological analogue, and hence there is no obvious definition of additivity. Therefore in [7] neither (A) nor (CA) was assumed.

5 GNS and modular construction

Given a differentiable manifold M , a collection $\{\mathcal{A}(\mathcal{O})\}_{\mathcal{O} \in M}$ of $*$ -algebras $\mathcal{A}(\mathcal{O})$ on bounded open sets $\mathcal{O} \in M$ is called a *net of $*$ -algebras*, iff isotony (N1) and covariance (N1) hold. The net is sometimes also denoted by $\mathcal{A} := \bigcup_{\mathcal{O}} \mathcal{A}(\mathcal{O})$. Selfadjoint elements of $\mathcal{A}(\mathcal{O})$ may be interpreted as possible measurements in \mathcal{O} . Two sets \mathcal{O}_1 and \mathcal{O}_2 , related by a topological isomorphism (e.g. a diffeomorphism) χ such that $\chi(\mathcal{O}_1) = \mathcal{O}_2$, may be identified straightforwardly only if there are no further obstructing relations between them. A relation like $\mathcal{O}_1 \subset \mathcal{O}_2$, in addition to the previous one, implies that \mathcal{O}_1 and \mathcal{O}_2 have to be considered as non-identical but topologically isomorphic sets. A similar situation holds on the level of algebras. Isotony (N1) in connection with covariance (N2) implies that $\mathcal{A}(\mathcal{O}_1)$ and $\mathcal{A}(\mathcal{O}_2)$ are isomorphic but non-identical algebras.

The state of a physical system is mathematically described by a positive linear functional ω on \mathcal{A} . Given the state ω , one gets via the GNS construction a representation π^ω of \mathcal{A} by a net of operator algebras on a Hilbert space \mathcal{H}^ω with a cyclic vector $\Omega^\omega \in \mathcal{H}^\omega$. The GNS representation $(\pi^\omega, \mathcal{H}^\omega, \Omega^\omega)$ of any state ω has an associated folium \mathcal{F}^ω , given as the family of those states $\omega_\rho := \text{tr} \rho \pi^\omega$ which are defined by positive trace class operators ρ on \mathcal{H}^ω .

Once a physical state ω (which implicitly contains all peculiarities of a particular observation) has been specified, one can consider in each algebra $\mathcal{A}(\mathcal{O})$ the equivalence relation

$$A \sim B \quad : \Leftrightarrow \quad \omega'(A - B) = 0, \quad \forall \omega' \in \mathcal{F}^\omega. \quad (5.1)$$

These equivalence relations generate a two-sided ideal

$$\mathcal{I}^\omega(\mathcal{O}) := \{A \in \mathcal{A}(\mathcal{O}) | \omega'(A) = 0\} \quad (5.2)$$

in $\mathcal{A}(\mathcal{O})$. The (dynamically relevant) state dependent algebra of observables $\mathcal{A}^\omega(\mathcal{O}) := \pi^\omega(\mathcal{A}(\mathcal{O}))$ may be constructed from the (kinematically relevant) algebra of observation procedures $\mathcal{A}(\mathcal{O})$ by taking the quotient

$$\mathcal{A}^\omega(\mathcal{O}) = \mathcal{A}(\mathcal{O}) / \mathcal{I}^\omega(\mathcal{O}). \quad (5.3)$$

The net of state-dependent algebras then is also denoted as $\mathcal{A}^\omega := \bigcup_{\mathcal{O}} \mathcal{A}^\omega(\mathcal{O})$. By construction, any diffeomorphism $\chi \in \text{Diff}(M)$ induces an algebraic isomorphism α_χ of the observation procedures. Nevertheless, for a given state ω , the action of α_χ will in general *not* leave \mathcal{A}^ω invariant. In order to satisfy

$$\alpha_\chi(\mathcal{A}^\omega(\mathcal{O})) = \mathcal{A}^\omega(\chi(\mathcal{O})). \quad (5.4)$$

the ideal $\mathcal{I}^\omega(\mathcal{O})$ must transform covariantly, i.e. the diffeomorphism χ must satisfy the condition

$$\alpha_\chi(\mathcal{I}^\omega(\mathcal{O})) = \mathcal{I}^\omega(\chi(\mathcal{O})) \quad (5.5)$$

for some algebraic isomorphism α_χ . Due to non-trivial constraints (5.5), the (dynamical) algebra of observables, constructed with respect to the folium \mathcal{F}^ω , does in general no longer exhibit the full $\text{Diff}(M)$ symmetry of the (kinematical) observation procedures. The symmetry of the observables is dependent on (the folium of) the state ω . Therefore, the selection of a folium of states \mathcal{F}^ω , induced by the actual choice of a state ω , results immediately in a breaking of the $\text{Diff}(M)$ symmetry. The diffeomorphisms which satisfy the constraint condition (5.5) form a subgroup called the *dynamical group* of the state ω . α_χ is called a *dynamical* isomorphism (w.r.t. the given state ω) w.r.t. χ , if (5.5) is satisfied.

The remaining dynamical symmetry group, depending on the folium \mathcal{F}^ω of states related to ω , has two main aspects which we have to examine in order to specify the physically admissible states: Firstly, it is necessary to specify its state dependent algebraic action on the net of observables. Secondly, one has to find a geometric interpretation for the group and its action on M .

If we consider the dynamical group as an *inertial*, and therefore global, manifestation of dynamically ascertainable properties of observables, then its (local) action should be correlated with (global) operations on the whole net of observables. This implies that at least some of the dynamical isomorphisms α_χ are not inner. (For the case of causal nets of algebras it was actually already shown that, under some additional assumptions, the isomorphisms of the algebras are in general not inner[13].)

In the following we consider instead of the net of observables $\mathcal{A}^\omega(\mathcal{O})$ the net of associated von Neumann algebras $\mathcal{R}^\omega(\mathcal{O})$.

Let us assume Ω^ω is a cyclic and separating vector for $\mathcal{R}_{\min}^\omega$ and $\mathcal{R}_{\max}^\omega$. By isotony it is so for any local von Neumann algebra $\mathcal{R}^\omega(\mathcal{O}_s^x)$ too. As a further consequence, on any region \mathcal{O}_s^x , the Tomita operator S and its conjugate F can be defined densely by

$$SA\Omega^\omega := A^*\Omega^\omega \quad \text{for } A \in \mathcal{R}^\omega(\mathcal{O}_s^x) , \quad (5.6)$$

$$FB\Omega^\omega := B^*\Omega^\omega \quad \text{for } B \in \mathcal{R}^\omega(\mathcal{O}_s^x)' . \quad (5.7)$$

The closed Tomita operator S has a polar decomposition

$$S = J\Delta^{1/2} , \quad (5.8)$$

where J is antiunitary and $\Delta := FS$ is the self-adjoint, positive modular operator. The Tomita-Takesaki theorem [14] provides us with a one-parameter group of state dependent isomorphisms α_t^ω on $\mathcal{R}^\omega(\mathcal{O}_s^x)$, defined by

$$\alpha_t^\omega(A) = \Delta^{-it} A \Delta^{it} , \quad \text{for } A \in \mathcal{R}_{\max}^\omega . \quad (5.9)$$

A strongly continuous unitary implementation of the modular group of the 1-parameter family of isomorphisms (5.9) w.r.t. ω is given by conjugate action of operators $e^{-it \ln \Delta}$, $t \in \mathbb{R}$. By (5.9) the modular group, for a state ω on the net of von Neumann algebras, defined by $\mathcal{R}_{\max}^\omega$, might be considered as a 1-parameter subgroup of the dynamical group. Note that, with Eq. (5.7), in general, the modular operator Δ is not located on \mathcal{O}_s^x . Therefore, in general, the modular isomorphisms (5.9) are not inner. The modular isomorphisms are known to act as inner isomorphisms, iff the von Neumann algebra $\mathcal{R}^\omega(\mathcal{O}_s^x)$ generated by ω contains only semifinite factors (type I and II), i.e. ω is a semifinite trace.

Above we considered concrete von Neumann algebras $\mathcal{R}^\omega(\mathcal{O}_s^x)$, which are in fact operator representations of an abstract von Neumann algebra \mathcal{R} on a GNS Hilbert space \mathcal{H}^ω w.r.t. a faithful normal state ω . In general, different faithful normal states generate different concrete von Neumann algebras and different modular isomorphism groups of the same abstract von Neumann algebra.

The outer modular isomorphisms form the cohomology group $\text{Out}\mathcal{R} := \text{Aut}\mathcal{R}/\text{Inn}\mathcal{R}$ of modular isomorphisms modulo inner modular isomorphisms. This group is characteristic for the types of factors contained in the von Neumann algebra [15]. Per definition $\text{Out}\mathcal{R}$ is trivial for inner isomorphisms. Factors of type III_1 yield $\text{Out}\mathcal{R} = \mathbb{R}$. This is a physically well-known case, realized by standard QFT, say on Minkowski space. Here a distinguished 1-parameter group of outer modular isomorphisms is given, which may be interpreted geometrically as a certain subgroup of the dynamical group.

6 Application for quantum gravity

On a region causally exterior to a topological horizon \mathcal{H} , on any d -dimensional spatial slice Σ , there exists a net of Weyl algebras for states with an *infinite* number of intersection points of edges and transversal $(d-1)$ -faces within any neighbourhood of the spatial boundary $\mathcal{H} \cap \Sigma \cong S^2$.

Σ be a spatial slice. C-causality constrains the algebras localized within Σ . On Σ it should hold

$$\mathcal{O}_1 \cap \mathcal{O}_2 = \emptyset \quad \Rightarrow \quad [\mathcal{A}(\mathcal{O}_1), \mathcal{A}(\mathcal{O}_2)] = 0. \quad (6.1)$$

A (spin network) state ω over the algebra $\mathcal{A}(\Sigma)$ may be defined by a closed, oriented differentiable graph γ embedded in Σ , with an infinite number of differentiable edges $e \in E$ intersected transversally by a differentiable $d-1$ -dimensional oriented surface S at a countable of intersection vertices $v \in V$. Let $C_\gamma \in \text{Cyl}$ be a \mathcal{C}^∞ Cylinder function with respect to an $\text{SO}(d)$ holonomy group on γ , i.e. on each closed *finite* subgraph $\gamma' \subset \gamma$ it is $C_{\gamma'} := c(g_1, \dots, g_N)$ where $g_k \in \text{SO}(d)$, and c is a differentiable function. With test function f the action of a derivation $X_{S,f}$ on Cyl is defined by

$$X_{S,f} \cdot C_\gamma := \frac{1}{2} \sum_{v \in V} \sum_{e_v \in E: \partial e_v \ni v} \kappa(e_v) f^i(v) X_{e_v}^i \cdot c, \quad (6.2)$$

where $\kappa(e_v) = \pm 1$ above/below S (for the following purposes we may just exclude the tangential case $\kappa(e_v) = 0$) and $X_{e_v}^i \cdot c$ is the action of the left/right invariant vector field (i.e. e_v is oriented away from/towards the surface S) on the argument of c which corresponds to the edge e_v . Let $\text{Der}(\text{Cyl})$ denote the span of all such derivations.

Here the classical (extended) phase space is the cotangent bundle $\Gamma = T^*\mathcal{C}$ over a space \mathcal{A} of (suitably regular) finitely localized connections. Let $\delta = (\delta_A, \delta_E) \in T_e\Gamma$. With suitable boundary conditions, a (weakly non-degenerate) symplectic form Ω over Γ acts via

$$\Omega|_{(A_e, E_e)}(\delta, \delta') := \frac{1}{\ell^2} \int_\Sigma \text{Tr} [*E \wedge A' - *E' \wedge A]. \quad (6.3)$$

After lifting from \mathcal{C} to Γ , the cylinder functions $q \in \text{Cyl}$ serve as (gauge invariant) classical elementary configuration functions on Γ . The derivations $p \in \text{Der}(\text{Cyl})$ serve as classical elementary momentum functions on Γ . They are obtained as the Poisson-Lie action of 2-dimensionally smeared duals of densitized triads E . $\text{Cyl} \times \text{Der}(\text{Cyl})$ has a Poisson-Lie structure

$$\{(q, p), (q', p')\} := (pq' - p'q, [p, p']), \quad (6.4)$$

where $[p, p']$ denotes the Lie bracket of p and p' . An antisymmetric bilinear form on $\text{Cyl} \times \text{Der}(\text{Cyl})$ is given by

$$\Omega_0((\delta_q, \delta_p), (\delta'_q, \delta'_p)) := \int_{\mathcal{C}_{\gamma \cup \gamma'} / \mathcal{G}_{\gamma \cup \gamma'}} d\mu_{\gamma \cup \gamma'} [pq' - p'q], \quad (6.5)$$

where $q, q' \in \text{Cyl}$ have support on γ resp. γ' , with $pq' - p'q \in \text{Cyl}$ integrable over $\mathcal{C}_{\gamma \cup \gamma'} / \mathcal{G}_{\gamma \cup \gamma'}$ with measure $d\mu_{\gamma \cup \gamma'}$.

On $T_e\Gamma$, the symplectic form Ω yields functions of the form $\Omega((\delta_A, \delta_E), \cdot)$. Canonical quantization then associates to any function $\Omega(f, \cdot)$ a selfadjoint operator $\hat{\Omega}(f, \cdot)$ and a corresponding unitary Weyl element $W(f) := e^{i\hat{\Omega}(f, \cdot)}$, both on some extended Hilbert space. With multiplication $W(f_1)W(f_2) := e^{i\Omega(f_1, f_2)}W(f_1 + f_2)$, and conjugation $*$: $W(f) \mapsto W(-f)$, the Weyl elements generate a $*$ -algebra. A norm on Γ is defined by $\|f\| := \frac{1}{4} \sup_{g \neq 0} \frac{\Omega(f, g)}{\langle g, g \rangle}$. The C^* -closure under the sup-norm then generates a C^* -algebra $CCR(W(f), \Omega)$. With regular Ω this CCR Weyl algebra is simple, i.e. there is no ideal. Observables of quantum 3-geometry are then the selfadjoint elements within a gauge and 3-diffeomorphism invariant C^* -subalgebra $\mathcal{A}_\gamma \subset C^*(W(f), f \in \Gamma)$. In a gauge and 3-diffeomorphism invariant representation of \mathcal{A}_γ , typical observables in are represented by configuration multiplication operators $C_\gamma \in \text{Cyl}$ on Hilbert space \mathcal{H}_γ , and by gauge-invariant and 3-diffeomorphism invariant combinations of derivative operators $X_{S, f} \in \text{Der}(\text{Cyl})$, like e.g. a certain quadratic combination which yields the area operator.

For each finite $\gamma' \subset \gamma$, the sets $E(\gamma')$ and $V(\gamma')$ of edges resp. vertices of γ' are finite. Then the connections $\mathcal{C}_{\gamma'} = \prod_{e \in E(\gamma')} G_e \cong G^{E(\gamma')}$ and the gauge group $\mathcal{G}_{\gamma'} = \prod_{v \in V(\gamma')} G_v \cong G^{V(\gamma')}$ on γ' inherit a unique measure from the measure on G (for compact G the Haar measure). The action of $\mathcal{G}_{\gamma'}$ on $\mathcal{C}_{\gamma'}$ is defined by $(gA)_e := g_{t(e)} A_e g_{s(e)}^{-1}$ where s and t are the source and target functions $E(\gamma') \rightarrow V(\gamma')$ respectively. This action gives rise to gauge orbits and a corresponding projection $\mathcal{C}_{\gamma'} \twoheadrightarrow \mathcal{C}_{\gamma'} / \mathcal{G}_{\gamma'}$. The projection induces the measure on $\mathcal{C}_{\gamma'} / \mathcal{G}_{\gamma'}$. Bounded functions w.r.t. to that measure define then the gauge invariant Hilbert space $\mathcal{H}_{\gamma'} := \mathcal{L}(\mathcal{C}_{\gamma'} / \mathcal{G}_{\gamma'})$.

However, over finite graphs, all is still QM rather than QFT. In order to obtain an infinite number of degrees of freedom on any finite localization domain which includes the inner boundary S^{d-1} (the intersection S^{d-1} of \mathcal{H} and Σ), let S^{d-1} be intersected by an infinite number of edges of some graph γ in the exterior spatial neighborhood of \mathcal{H} . In the 3+1-dimensional case, evaluation of the area operator on the puncture of the boundary S^2 from edge p yields a quantum of area proportional to $j_p(j_p + 1)$ for edge p carrying a spin- j_p representation of the group G . Since S^2 is compact the punctures should have at least one accumulation point. Hence for typical configurations in the principal representation, near that accumulation point the area will explode to infinity. When almost all punctures

are located in arbitrary small neighborhoods of a finite number n of accumulation points, corresponding states represent quantum geometries of a black hole with n stringy hairs extending out to infinity. In particular, the $n = 1$ case was discussed in more detail in [16].

Acknowledgment

I would like to thank for the hospitality of the Instituto de Matematicas y Fisica Fundamental at CSIC Madrid, where part of the manuscript was completed.

References

- [1] K. Osterwalder and R. Schrader, Axioms for Euclidean Green's functions. I, *Commun. Math. Phys.* **31**, 83-112 (1973); II, *Commun. Math. Phys.* **42**, 281-305 (1975).
- [2] J. Glimm and A. Jaffe, Quantum Physics, 2nd. ed., (Springer, New York, 1987).
- [3] A. Ashtekar, D. Marolf, J. Mouro, and T. Thiemann , Osterwalder-Schrader Reconstruction and Diffeomorphism Invariance quant-ph/9904094
- [4] K.-H. Rehren , Algebraic Holography , <http://xxx.lanl.gov/abs/hep-th/9905179>
- [5] J. M. Maldacena , The Large N Limit of Superconformal Field Theories and Supergravity , *Adv. Theor. Math. Phys.* **2**, 231-252 (1998).
- [6] E. Witten , Anti de Sitter Space and Holography , *Adv. Theor. Math. Phys.* **2**, 253-291 (1998).
- [7] M. Rainer, The Role of Dilations in Diffeomorphism Covariant Algebraic Quantum Field Theory, gr-qc/9710081.
- [8] M. Rainer and H. Salehi, A Regularizing Commutant Duality for a Kinematically Covariant Partial Ordered Net of Observables, gr-qc/9708059.
- [9] M. Rainer, General Regularized Algebraic Nets for General Covariant QFT on Differentiable Manifolds, gr-qc/9705084.
- [10] U. Bannier, *Commun. Math. Phys.* **118**, 163 (1988).
- [11] U. Bannier, *Int. J. Theor. Phys.* **33**, 1797 (1994).
- [12] M. Rainer, Cones and Causal Structures on Topological and Differentiable Manifolds, gr-qc/9905106.
- [13] Wollenberg M. 1990, On Causal Nets of Algebras, pp. 337-344 in: *Operator Theory, Advances and Applications* **43**, *Linear Operators in Function Spaces, Proc. 12th Int. Conf. on Operator Theory*, Timișoara, Romania (June 6-16, 1988), eds.: H. Helson, B. Sz.-Nagy, and F.-H. Vasilescu.
- [14] R. Haag, Local Quantum Physics (Springer Verlag, Berlin, 1992).
- [15] A. Connes, Noncommutative Geometry (Academic Press, N.Y., 1995).
- [16] M. Rainer, On the fundamental length of quantum geometry and the black hole entropy, gr-qc/9903091.